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Electroweak Symmetry on the Tangent Bundle

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Various symmetries of elementary particles can be represented by gauge transformations acting on a fiber of the tangent bundle. These are diffeomorphisms of linear groups which act on vertical vector fields. It is shown how the electroweak vector boson potentials and a corresponding Kaluza–Klein-like metric can be obtained by application of SU(2) × U(1) to a tangent fiber. This geometry gives a more unified approach to gravitation and gauge symmetries.

1. INTRODUCTION

Consider the four-dimensional base manifold M of space-time with tangent bundle Π: $TM \rightarrow M$. The bundle TM is itself an eight-dimensional differentiable manifold with its own tangent structure TTM which can be separated into horizontal and vertical subspaces. In the simplest case, a Minkowski metric can be associated with the horizontal part and a metric on the vertical part can also be taken initially to be Minkowskian. This means there is a mapping of an orthonormal basis of M to a similar basis on the fiber of TM given by an orthonormal tetrad. The tetrad accomplishes a trivial soldering of the fiber to the base space. Physically, one can think of the coordinates of M as "laboratory" coordinates and the fiber coordinates as "internal" coordinates of a local system, say, a particle. The tetrad is simply a Lorentz transformation. Starting from this geometry, it is possible to operate on the fiber coordinates with a linear transformation without at the same time transforming the base space coordinates. This is a vertical automorphism, sometimes called a "pure" gauge transformation [5, 25, 31]. The transformation group could be GL(4; R) or one of its subgroups. The effect of such a pure gauge transformation is to rotate or distort the vertical tangent basis. If the fiber metric is not preserved, then the base space metric also changes

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due to the soldering. Thus, even though the coordinates of M are unaffected, M receives a new metric. This idea will be made clearer in the next section.

In general, for a pure gauge transformation dependent on the base space coordinates x, the new metric of M is also x dependent. This could produce any of the metrics of general relativity. Equivalently, these same metrics can be associated with general covariant coordinate transformations acting on M in the traditional way. (A coordinate transformation of this type would be the inverse of the corresponding transformation to a local inertial frame.) The local equivalence of coordinate and gauge transformations has been discussed [9, 10, 13]. The use of this equivalence illuminates the problem of general covariance versus gauge invariance (as articulated, recently, for example, by Wilczek [35]). Note, incidentally, that even though the metrics associated with the gauge and coordinate transformations may be the same, the connections are different, being the affine connection and the Levi-Civita connection, respectively [9, 13].

Even more generally, if the gauge transformation is also dependent on the coordinates y of the tangent fiber, the result can be a Finsler or Lagrange metric [27]. Actually, the theory is best expressed in the context of Finsler or Lagrange geometry. The reader unfamiliar with this mathematics should consult one or more of the recent treatments [2, 4, 27]. For example, the gauge transformations described above are closely related to certain transformations of differential forms exploited by Chern and collaborators (ref. 14, p. 53; ref. 6, p. 17).

There are several ways this pure gauge transformation concept can be applied: For example, the tetrad could be a moving frame [11, 12] and the group could be O(1, 3; R), a Lorentz group with gauge parameters being functions of a path parameter. This produces well-known types of particle transport such as Frenet–Serret and Fermi–Walker. Another example is the Abelian group U(1). This has been shown in the context of Finsler geometrical methods to produce a metric which has the appearance of a Kaluza–Klein type of metric, but a somewhat different interpretation. This leads to a new approach to the unification of gravitation and electromagnetism [10, 13]. In this theory, the potentials are related to the tetrads themselves and are not connections, as in the usual treatments.

In previous work [10–12], it was mentioned that the idea of tangent gauge transformations would also work for symmetries such as SU(2) or the electroweak SU(2) $\times$ U(1). Indeed, the approach is compatible with any subgroup of GL(4; R). It is the purpose of this paper to present a specific representation of the electroweak symmetry applied to tangent fibers and to show how this relates to the Weinberg–Salam theory. This provides the SU(2) solution of Open Problem #2 [13]. Note that, although work in this paper is done in a four-dimensional tangent fiber, many of the results can be translated to fibers of other dimension corresponding to well-known Yang–Mills theories. This is accomplished by mappings given in refs. 10 and 29.

2. GEOMETRICAL PRELIMINARIES

Local coordinates in the eight-dimensional TM are designated (x^μ, y^μ). The natural coordinate vector fields are $\partial/\partial x^\mu$ and $\partial/\partial y^\mu$. Note that the y's are both the fiber coordinates y^μ and the components of the tangent vector $\mathbf{y} = y^\mu\,(\partial/\partial x^\mu)$.

The transformation which is a change of local section or coordinate transformation

$$x'^\mu = x'^\mu(x^\nu) \tag{1}$$

will be examined following Miron and Anastasiei (ref. 27, Chapters VI and VII). This will be contrasted presently with the pure gauge transformations described in the Introduction. The coordinate transformation matrix is written as $X^{*\mu}_\nu = \partial x'^\mu/\partial x^\nu$.

Since $dx'^\mu = X^{*\mu}_\nu\, dx^\nu$, the identification of the y^μ as tangent vector components implies

$$y'^\mu = X^{*\mu}_\nu y^\nu \tag{2}$$

The natural basis of vector fields transforms as

$$\frac{\partial}{\partial x'^\mu} = X^\nu_\mu \frac{\partial}{\partial x^\nu} + \frac{\partial X^\nu_\mu}{\partial x'^\lambda} y'^\lambda \frac{\partial}{\partial y^\nu} \quad \frac{\partial}{\partial y'^\mu} = X^\nu_\mu \frac{\partial}{\partial y^\nu} \tag{3}$$

with $X^\mu_\nu X^{*\nu}_\lambda = \delta^\mu_\lambda$.

The basis obviously does not transform covariantly. A useful procedure at this point is to construct a new basis, called the local adapted basis, $(\delta/\delta x^\mu, \partial/\partial y^\mu)$,

$$\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^\nu_\mu \frac{\partial}{\partial y^\nu} \tag{4}$$

This transforms covariantly if N transforms according to

$$N'^\nu_\mu = X^{*\nu}_\lambda N^\lambda_\rho X^\rho_\mu + X^{*\nu}_\lambda \frac{\partial X^\lambda_\mu}{\partial x'^\rho} y'^\rho \tag{5}$$

N is called the nonlinear connection.

A dual basis for the adapted basis is $(dx^\mu, \delta y^\mu)$, with

$$\delta y^\mu = dy^\mu + N^\mu_\nu\, dx^\nu \tag{6}$$

The nonlinear connection defines a horizontal lift which splits TTM into horizontal and vertical subspaces.

A metric on *TM* is

$$dS^2 = G_{\mu\nu}dx^\mu dx^\nu + H_{\mu\nu}\delta y^\mu \delta y^\nu \tag{7}$$

This metric is block diagonal in 4+4 dimensions in the adapted basis and is scalar under a coordinate transformation. This is sometimes called the horizontal lift metric [15].

Traditional Kaluza–Klein theories, however, consider a different metric associated with the natural basis (dx^μ, dy^μ). The use of (6) in (7) gives the familiar matrix

$$\left\| \begin{matrix} G_{\mu\nu} + N^\rho_\mu N^\sigma_\nu H_{\rho\sigma} & N^\rho_\mu H_{\rho\sigma} \\ N^\sigma_\nu H_{\rho\sigma} & H_{\rho\sigma} \end{matrix} \right\| \tag{8}$$

Here the vertical metric is four dimensional, but the generalization to any number of dimensions is obvious. See, for example, ref. 10.

Kaluza–Klein theory is developed from the horizontal block of the metric. There are a number of recognized problems with this metric [20]. One problem is the presence of the connection *N* in the metric. This introduces a difficult nonlinearity into the theory which has no reasonable physical interpretation. How can a metric contain a connection? How does one interpret a new connection computed from this metric?

This problem does not arise when the adapted basis is retained. The metric components *G* do not involve the connection. Both *G* and *H* transform covariantly, which implies that the physical properties of space-time are preserved under a coordinate transformation, as should be expected. The equivalence principle of general relativity is applicable.

The use of the adapted basis leads to reasonable physical equations. For example, a geodesic equation is easily derived by standard methods. The connection *N* is related to the Levi-Civita connection, which in turn is related to the external field. The equation of motion is thus a geodesic equation rather than an equation of geodesic deviation as in the usual Kaluza–Klein theories.

Attention is now directed to the pure gauge transformation. For physical reasons, it is assumed that initially both the horizontal metric *G* and the vertical metric *H* are Minkowskian (i.e., a Lorentz type, $\eta_{\mu\nu}$). This leaves out purely gravitational effects and allows a focus on the physical results of the gauge transformation. As indicated in the Introduction, the *G* components can refer to a laboratory frame and the *H* components can describe the internal space of a local system such as a particle. The vector **y** would then be the velocity of the particle.

The total metric is

$$dS^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu + \eta_{\alpha\beta}\, dy^\alpha\, dy^\beta \tag{9}$$

Early Greek indices are now used to distinguish the vertical or internal subspace.

Obviously there is a trivial soldering or mapping between the two metrics:

$$\eta_{\mu\nu} = \eta_{\alpha\beta} e^{\alpha}_{\mu} e^{\beta}_{\nu} \tag{10}$$

The orthonormal tetrads e^{α}_{μ} are just Lorentz transformations. The μth column of the tetrad represents the components of a basis for the lab frame expressed in the basis of the internal frame.

The tetrad soldering or coupling of the fiber space to the base space has occasionally been advanced [18, 20, 25, 33] as an ingredient of a gauge theory of gravitation.

The pure gauge transformation is now defined by

$$\bar{x}^{\mu} = x^{\mu}, \qquad \bar{y}^{\alpha} = V^{*\alpha}_{\beta} y^{\beta} \tag{11}$$

In other words, it acts only on the fiber coordinates and not on the coordinates of M. This is the same transformation defined by Miron and Anastasiei (ref. 27, p. 97), Nash and Sen (ref. 28, 177), Mack (ref. 25, p. 141), and many others.

The transformation matrix V^* is nonsingular and defines a diffeomorphism such that the fiber is preserved. It is a finite representation which contains the parameters of the group.

The vertical basis map is

$$\frac{\partial}{\partial \bar{y}^{\alpha}} = V^{\gamma}_{\alpha} \frac{\partial}{\partial y^{\gamma}}, \qquad V^{\gamma}_{\alpha} V^{*\alpha}_{\beta} = \delta^{\gamma}_{\beta} \tag{12}$$

The new internal metric is

$$H_{\alpha\beta} = \eta_{\gamma\delta} V^{\gamma}_{\alpha} V^{\delta}_{\beta} \tag{13}$$

Various physical effects of this transformation are possible, depending on the group associated with V. Examples are O(1, 3), U(1), and SU(2), as mentioned in the Introduction. In general, a Lorentz group is metric preserving and could correspond to particle transport (boosts and rotations). The other groups, while metric preserving when applied to their own principal bundles, are not metric preserving as applied here to the tangent space. They are similar in this respect to, for example, “shear” and “scale” transformations in the terminology of Hehl *et al.* [18] and Ivanenko and Sardanashvily [20].

The soldering mechanism commutes with the gauge transformation, so that the new base space metric is

$$G_{\mu\nu} = H_{\alpha\beta} e^{\alpha}_{\mu} e^{\beta}_{\nu} \tag{14}$$

The metric G can also be written in the form

$$G_{\mu\nu} = \eta_{\gamma\delta} b^{\gamma}_{\mu} b^{\delta}_{\nu} \tag{15}$$

where

$$b^{\alpha}_{\mu} = V^{\alpha}_{\beta} e^{\beta}_{\mu} \tag{16}$$

is a new tetrad (not necessarily orthonormal) which represents the components of a new set of bases for *M*.

So the gauge transformation acting in the tangent fiber gives not only a new vertical subspace metric, but also a new metric in the base space. This new metric in *M* occurs without a change of coordinates. One could also consider a more general gauge transformation where *V* is dependent on *y*. This leads to metrics which are *y* dependent and are either Finsler metrics or generalized Lagrange metrics [27].

In contrast with many gauge theories, where the internal symmetry is only loosely attached, this theory is very much like some theories of gravitation, especially tetrad theories, with a close coupling between laboratory space-time and the internal particle space.

3. ELECTROMAGNETISM FROM U(1)

As a simple example of how this works, the U(1) case is considered first. It will be useful to write the transformation matrix in the form

$$V^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + \psi^{\alpha}_{\beta} \tag{17}$$

The new metric is

$$G_{\alpha\beta} = \eta_{\alpha\beta} + 2\Psi_{\alpha\beta} + \eta_{\gamma\delta}\Psi^{\gamma}_{\alpha}\Psi^{\delta}_{\beta} = \eta_{\alpha\beta} + U_{\alpha\beta} \tag{18}$$

where $U_{\alpha\beta}$ is a metric-like matrix containing the parameters of the gauge group.

In laboratory coordinates, the metric is

$$G_{\mu\nu} = \eta_{\mu\nu} + U_{\alpha\beta} e^{\alpha}_{\mu} e^{\beta}_{\nu} \tag{19}$$

A particular example of gauge transformation which leads to electromagnetism is given by

$$V^{\alpha}_{\beta} = \mathrm{diag}(1 + \psi, 1, 1, 1) \tag{20}$$

This has the form (17) where the only nonzero element of ψ^{α}_{β} is $\psi^{0}_{0} = \psi$.

This is a noncompact representation of the one-parameter Abelian group. It is related to the usual compact representation, U(1), which is also one-parameter Abelian. The actual symmetry of electromagnetism can be considered to be either compact or noncompact (ref. 21, p. 101).

The metric (19) is easily seen to be

$$G_{\mu\nu} = \eta_{\mu\nu} + (2\psi + \psi^2)e^0_\mu e^0_\nu \tag{21}$$

which can be reparametrized using $(1 + \Psi)^2 = 1 + \chi$ to

$$G_{\mu\nu} = \eta_{\mu\nu} + \chi e^0_\mu e^0_\nu \tag{22}$$

But this is just a version of the metric

$$G_{\mu\nu} = \eta_{\mu\nu} + kB_\mu B_\nu \tag{23}$$

where k is proportional to the gravitational constant and B_μ is gauge related to the electromagnetic potential vector of an external field. Essentially, the gauge transformation has "turned on" the electromagnetic field. The metric (23) was introduced in a Finsler context [7] and has since been studied by several investigators [1, 4, 19, 22, 24, 26, 32]. This is indeed an electromagnetic metric since it produces a geodesic equation which is the Lorentz charged particle equation and a curvature which contains the electromagnetic energy-momentum tensor [7, 13]. Note that, in contrast with traditional Kaluza–Klein treatments, where potentials are connections and fields are curvatures, here the potential appears in the metric as a component of a tetrad and the fields are part of the connection. This means, for example, that the equations of motion are geodesic equations instead of equations of geodesic deviation. Fundamentally, in the traditional version, the natural bundle basis is taken and (23) lives in the total bundle space. Here, the local adapted basis is taken and (23) lives only in the base space.

The B_μ are just vectors and transform as vectors under coordinate transformations. Consequently, the electromagnetic fields, $F_{\mu\nu} = \partial B_\nu/\partial x^\mu - \partial B_\mu/\partial x^\nu$, transform as tensors, as they should. The connections are related to these potentials and fields by equations given, for example, in ref. 13 and transform properly as connections.

4. ELECTROWEAK SYMMETRY

The Weinberg–Salam theory involves the interaction of spinor lepton fields with four vector gauge bosons. The boson potentials are written here as W^0_μ, W^1_μ, W^2_μ, W^3_μ. The potential W^0_μ is commonly labeled B_μ; the notation is changed here to avoid identification with the B_μ of the preceding section.

These boson potentials can be derived from a U(1) (actually a noncompact, one-parameter Abelian) gauge transformation of the form

$$V^\alpha_\beta = \text{diag}(1 + \rho, 1 + \mu, 1 + \mu, 1 + \mu) \tag{24}$$

The parameters ρ and μ are related by a constant factor, as will be seen, so only a single Abelian parameter is involved.

The new basis is

$$b_\mu^\alpha = ((1 + \rho)e_\mu^0, (1 + \mu)e_\mu^1, (1 + \mu)e_\mu^2, (1 + \mu)e_\mu^3)^T \tag{25}$$

with metric

$$H_{\alpha\beta} = \mathrm{diag}((1 + R), -(1 + M), -(1 + M), -(1 + M)) \tag{26}$$

where $R = 2\rho + \rho^2$ and $M = 2\mu + \mu^2$. The laboratory metric (19) is

$$G_{\mu\nu} = \eta_{\mu\nu} + Re_\mu^0 e_\nu^0 - M(e_\mu^1 e_\nu^1 + e_\mu^2 e_\nu^2 + e_\mu^3 e_\nu^3) \tag{27}$$

This suggests the identification

$$W_\mu^0 = R^{1/2} e_\mu^0, \qquad W_\mu^i = M^{1/2} e_\mu^i \tag{28}$$

Other schemes for setting the boson vectors are possible, but (28) appears to be the simplest and gives a good picture of the method. The metric then becomes

$$G_{\mu\nu} = \eta_{\mu\nu} + \eta_{\alpha\beta} W_\mu^\alpha W_\nu^\beta \tag{29}$$

which has the appearance of a classic Kaluza–Klein Yang–Mills expression. Again, however, the W's are potentials, but not connections.

Consider now the SU(2) part of the symmetry. The most natural representation of SU(2) as a 4×4 matrix uses the quaternion covering group SL(1; Q). A useful representation of one rotation is

$$V_\alpha^\beta = \left\Vert \begin{matrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{matrix} \right\Vert \tag{30}$$

The associated metric is easily computed from (13) and (14):

$$\begin{aligned} G_{\mu\nu} = {} & \eta_{\mu\nu} + (\cos^2\theta - \sin^2\theta - 1)e_\mu^0 e_\nu^0 - 2\sin\theta\cos\theta(e_\mu^0\, e_\nu^3 + e_\mu^3\, e_\nu^0) \\ & - (\cos^2\theta - \sin^2\theta - 1)e_\mu^3 e_\nu^3 \end{aligned} \tag{31}$$

This result shows that the SU(2) group is not entirely metric preserving on the tangent space. The e_μ^1, e_μ^2 portion of the metric is preserved and is a simple space rotation.

The e_μ^0, e_μ^3 portion of the metric, as is obvious from (31), is not preserved and experiences a mixing of space and time components. The physical interpretation of this is easily seen by the application of (30) to the $(W_\mu^0, 0, 0, W_\mu^0)^T$ state, using (28), which gives the mixed potentials

$$A_\mu = W_\mu^0 \cos\theta - W_\mu^3 \sin\theta, \qquad Z_\mu = W_\mu^0 \sin\theta + W_\mu^3 \cos\theta \tag{32}$$

For θ equal to the Weinberg angle, the potential Z_μ is the massive neutral boson and A_μ is the potential of the massless electromagnetic field. Since

$$A_\mu A^\mu = R \cos^2\theta - M \sin^2\theta = 0 \tag{33}$$

then $R^{1/2} = M^{1/2} \tan\theta$, giving the relative strength of the Abelian gauge parameters. This reproduces the ratio of the weak and electromagnetic coupling constants.

A Kaluza–Klein-like expression like (19) of this metric can easily be obtained using (24) and (30). Metrics involving the other two rotations or various combinations of SU(2) symmetries are also derivable in a straightforward fashion.

So the electromagnetic potential is gauge related to the timelike component of the basis tetrad. The actual gauge transformation is an Abelian dilation of the timelike component which produces a change of scale in the spacetime metric. The parameters of the previous section can be related by setting $\chi = R$. The potentials then satisfy $W^0_\mu = k^{1/2} B_\mu$.

The weak potentials W^i_μ involve a dilation of the spacelike components accompanied by a similar change of scale. This indicates a possible more fundamental interpretation of gauge transformations as distortions of spacetime itself.

Note that there is no need for compactification or dimensional reduction in this theory since the "extra" four dimensions have a natural physical interpretation as being related to the system velocity.

The appropriate connections will be those computed from the metric (29) and not the usual connections involving the potentials themselves. The new connections contain derivatives of the potentials and/or fields as already worked out for the electromagnetic case [7, 9, 13].

In the complete electroweak theory, the boson potentials are coupled to spinor lepton solutions. The physical interpretation of related symmetries of the internal space in spinor geometry is a topic for future investigation.

4. DISCUSSION

The symmetries of the boson gauge potentials of the Weinberg–Salam model (ref. 21, p. 337; ref. 34, p. 307) have been reproduced by a pure gauge transformation on the tangent bundle. In the process, an alternate geometrical foundation for gauge theory has been pointed out. Principal features of this foundation are the coupling of internal and external space metrics and a shifting of the traditional correspondence of potential ↔ connection, field ↔ curvature to a new correspondence of potential ↔ tetrad (a part of the metric), field ↔ connection [7, 9–11, 13].

Since the new correspondence is the same as in many gauge theories of gravitation, a promising avenue toward unified theories has been opened. For example, much effort has been expended trying to fit gravitation into the traditional correspondence scheme. However, as pointed out by Ivanenko and Sardanashvily (ref. 20, p. 32), gravitational tetrad potentials can simply not be connections. In the new scheme, both gravitational and gauge potentials appear in the metric, while both gravitational and gauge fields occur in the connection. The equation of motion is a geodesic equation and not an equation of geodesic deviation.

Another key feature is the contrast between coordinate and pure gauge transformations. The coordinate transformations produce the nonlinear connections N (and their associated Levi-Civita connections) which relate to "transplantations" between neighboring points in space-time. The pure gauge transformations produce affine connections [9, 11, 13] which relate to the transport of a particle under the gauge field. There is a correlation of the internal potential of the particle with the external potential as discussed by Konopinski [23], Schweizer [30], and Beil [8].

The manner in which coordinate and pure gauge transformations can be combined has been given in ref. 11. Total particle transport is determined by a Levi-Civita connection plus a pullback of the gauge connection under a coordinate transformation. This gives a naturally unified approach to gravitational and gauge field effects. There is a general equivalence between gravitational and gauge connections.

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REFERENCES

1. M. Anastasiei and H. Shimada (1998). *Balkan J. Geom.* **3**, 1.
2. P. L. Antonelli, R. S. Ingarden and M. Matsumoto (1993). *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology* (Kluwer, Dordrecht).
3. G. S. Asanov (1985). *Finsler Geometry, Relativity and Gauge Theories* (Reidel, Dordrecht).
4. G. S. Asanov and T. Kawaguchi (1990). *Tensor N. S.* **49**, 99.
5. M. F. Atiyah, N. Hitchin, and E. M. Singer (1987). *Proc. R. Soc. Lond. A* **362**, 425.
6. D. Bao, S. S. Chern, and Z. Shen (1996). In *Finsler Geometry*, D. Bao, S. S. Chern, and Z. Shen, eds. (American Mathematical Society, Providence, Rhode Island).
7. R. G. Beil (1987). *Int. J. Theor. Phys.* **26**, 189.
8. R. G. Beil (1991). *Int. J. Theor. Phys.* **30**, 1663.
9. R. G. Beil (1992). *Int. J. Theor. Phys.* **31**, 1025.
10. R. G. Beil (1993). *Int. J. Theor. Phys.* **32**, 1021.
11. R. G. Beil (1995a). *Found. Phys.* **25**, 717.
12. R. G. Beil (1995b). *Tensor N. S.* **56**, 175.
13. R. G. Beil (1996). In *Finsler Geometry*, D. Bao, S. S. Chern, and Z. Shen, eds. (American Mathematical Society, Providence, Rhode Island).
14. S. S. Chern (1996). In *Finsler Geometry*, D. Bao, S. S. Chern and Z. Shen, eds. (American Mathematical Society, Providence, Rhode Island).

15. Y. M. Cho (1975). *J. Math. Phys.* **16**, 2029.
16. T. Fukuyama (1988). *Gen. Rel. Grav.* **20**, 89.
17. R. Gilmore (1974). *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York).
18. F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman (1989). *Found. Phys.* **19**, 1075.
19. R. S. Ingarden, M. Matsumoto, and L. Tamassy (1997). *Nonlinear World* **4**, 85.
20. D. Ivananko and G. Sardanashvily (1983). *Phys. Rep.* **94**, 1.
21. M. Kaku (1993). *Quantum Field Theory* (Oxford University Press, Oxford).
22. T. Kawaguchi and R. Miron (1989). *Tensor N.S.* **48**, 52.
23. E. J. Konopinski (1978). *Am. J. Phys.* **46**, 499.
24. J. Lawrynowicz, W. A. Rodrigues, and L. Wojtczak (1996). *Bull. Soc. Sci. Lett. Lodz* **46**, 57.
25. G. Mack (1981). *Fortschr. Physik* **29**, 135.
26. R. Miron and T. Kawaguchi (1991). *Int. J. Theor. Phys.* **30**, 1521.
27. R. Miron and M. Anastasiei (1994). *The Geometry of Lagrange Spaces: Theory and Applications* (Kluwer, Dordrecht).
28. C. Nash and S. Sen (1983). *Topology and Geometry for Physicists* (Academic Press, London).
29. N. Rosen and G. E. Tauber (1984). *Found. Phys.* **14**, 171.
30. M. A. Schweizer (1990). *Am. J. Phys.* **58**, 930.
31. M. Socolovsky (1991). *J. Math. Phys.* **32**, 2522.
32. P. C. Stavrinos, D. Nikolopoulos, and N. Prezas (1997). In *Proceedings of the IV and V Romanian National Conferences on Theoretical Physics, General Relativity and Gravitation, Volumes 4–5, 1994–1995* (Bistrita, Romania).
33. A. Trautman (1980). In *General Relativity and Gravitation*, Vol. 1, A. Held, ed. (Plenum Press, New York).
34. S. Weinberg (1996). *The Quantum Theory of Fields*, Vol. II (Cambridge University Press, Cambridge).
35. F. Wilczek (1998). *Phys. Rev. Let.* **80**, 4851.